

LECTURE 1: PERIODIC FUNCTIONS, FOURIER SERIES AND HALF – RANGE SERIES

1.0 PERIODIC FUNCTIONS

A function $f(x)$ is said to be periodic if there exists a number $T > 0$ such that $f(x + T) = f(x)$ for every x . The T is called the period of $f(x)$.

Intuitively, periodic functions have repetitive behavior.

A periodic function can be defined as a finite interval, then copied and pasted so that it repeats itself.

Some examples of periodic functions are:-

- With a period 2π , $\sin x$ and $\cos x$ are periodic
- With a period of 2, $\sin(\pi x)$ and $\cos(\pi x)$ are periodic
- If L is a fixed number, then $\sin\left(\frac{2\pi x}{L}\right)$ and $\cos\left(\frac{2\pi x}{L}\right)$ have period L

Note: sine and cosine are the most basic periodic functions.

1.1 FOURIER SERIES

Let $q > 0$ be a fixed number and $f(x)$ be a periodic function with period $2q$, defined on $(-q, q)$. The Fourier series of the function $f(x)$ is a way of expanding the function $f(x)$ into an infinite series involving sines and cosines as shown below:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{q}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{q}\right)$$

Where a_0 , a_n and b_n are known as the fourier coefficients of $f(x)$ and are expressed by the formulas below:

$$a_0 = \frac{1}{q} \int_{-q}^q f(x) dx$$

$$a_n = \frac{1}{q} \int_{-q}^q f(x) \cos\left(\frac{n\pi x}{q}\right) dx$$

$$b_n = \frac{1}{q} \int_{-q}^q f(x) \sin\left(\frac{n\pi x}{q}\right) dx$$

Things to Note

- ✓ It is okay to calculate the integrals that give the coefficients a_0 , a_n and b_n and insert them back into the series formula, to find the Fourier series.
- ✓ Basically, the function $f(x)$ is defined piecewise.
- ✓ The Fourier series has a major advantage over Taylor series since it can be applied or used to solve functions that have discontinuities.

1.2 USEFUL IDENTITIES FOR FOURIER SERIES

If n is an integer, then:

$$\sin(n\pi) = 0. \text{ For example, } \sin(\pi) = \sin(2\pi) = \sin(3\pi) = \sin(10\pi) = 0$$

For cosine, $\cos(n\pi) = (-1)^n = \begin{cases} 1 & \text{even } n \\ -1 & \text{odd } n \end{cases}$. For example, $\cos(\pi) = \cos(3\pi) = \cos(7\pi) = -1$ and $\cos(0\pi) = \cos(2\pi) = \cos(6\pi) = 1$.

1.3 FOURIER COEFFICIENTS OF AN EVEN FUNCTION

For an even function, the formulas for the coefficients are simplified. To be precise, if $f(x)$ is even, $f(x)\sin\left(\frac{n\pi x}{q}\right)$ is thus an odd function.

$$\text{Therefore, } b_n = \frac{1}{q} \int_{-q}^q \underbrace{f(x)}_{\text{even}} \underbrace{\sin\left(\frac{n\pi x}{q}\right)}_{\text{odd}} dx = 0$$

From the above expression, it can be deduced that for even functions, b_n coefficients are all 0.

1.4 FOURIER COEFFICIENTS OF AN ODD FUNCTION

If the function $f(x)$ is odd, the a_0 and a_n becomes:

$$a_0 = \frac{1}{q} \int_{-q}^q \underbrace{f(x)}_{\text{odd}} dx = 0$$

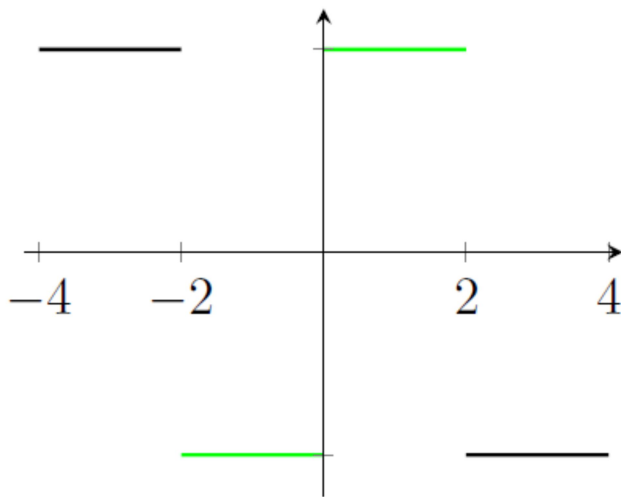
$$a_n = \frac{1}{q} \int_{-q}^q \underbrace{f(x)}_{\text{odd}} \underbrace{\cos\left(\frac{n\pi x}{q}\right)}_{\text{even}} dx = 0$$

Note: There could be situations where the function is neither even nor odd. In a situation like this, the original formula is used to compute the Fourier coefficients.

QUESTION 1

Find the Fourier series for the function below:

$$f(x) = \begin{cases} -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$$



SOLUTION

Recall that,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{q}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{q}\right)$$

From the above equation, it can be seen that $f(x)$ is an odd function. Therefore, $a_0 = a_n = 0$ for each n .

For b_n ,

$$b_n = \frac{1}{q} \int_{-q}^q f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$b_n = \frac{1}{2} \int_{-2}^0 (-1) \sin\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2} \int_0^2 (1) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$\begin{aligned}
&= \frac{1}{2} \left[\frac{2 \cos\left(\frac{n\pi x}{2}\right)}{n\pi} \right]_{-2}^0 + \frac{1}{2} \left[\frac{-2 \cos\left(\frac{n\pi x}{2}\right)}{n\pi} \right]_0^2 \\
&= \left[\frac{\cos\left(\frac{n\pi x}{2}\right)}{n\pi} \right]_{-2}^0 + \left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{n\pi} \right]_0^2 \\
&= \frac{1}{n\pi} \left[\cos\left(n \frac{\pi}{2} \cdot 0\right) - \cos(-n\pi) \right] + \frac{1}{n\pi} \left[-\cos(n\pi) - (-\cos(n \frac{\pi}{2} \cdot 0)) \right] \\
&= \frac{1}{n\pi} [\cos(0) - \cos(-n\pi)] + \frac{1}{n\pi} [-\cos(n\pi) - (-\cos(0))] \\
&= \frac{1}{n\pi} [1 - (-1)^n] + \frac{1}{n\pi} [-(-1)^n - (0)] = \frac{2}{n\pi} [1 - (-1)^n]
\end{aligned}$$

The Fourier series for the function becomes:

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - (-1)^n) \sin\left(\frac{n\pi x}{2}\right) \\
&= \underbrace{\frac{4}{\pi} \sin\left(\frac{\pi x}{2}\right)}_{n=1} + \underbrace{\frac{4}{3\pi} \sin\left(\frac{3\pi x}{2}\right)}_{n=3} + \dots
\end{aligned}$$

Note: It will be observed that the even b_n terms are all 0.

QUESTION 2

Find the Fourier series for the function shown below. The cycle with period 2π ranges from $x = -\pi$ to $x = \pi$.

$$f(x) = \begin{cases} 0 & -\pi < x < \frac{\pi}{2} \\ 6 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases},$$

SOLUTION

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{q}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{q}\right)$$

$$a_0 = \frac{1}{q} \int_{-q}^q f(x) dx = 0$$

Where $q = \pi$.

For the function,

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^{-\frac{\pi}{2}} (0) dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (6) dx + \int_{\frac{\pi}{2}}^{\pi} (0) dx \right]$$

$$= \frac{1}{\pi} \left[6\pi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$a_0 = 6$$

To find a_n :

$$a_n = \frac{1}{q} \int_{-q}^q f(x) \cos\left(\frac{n\pi x}{q}\right) dx$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{-\frac{\pi}{2}} (0) \cos nx dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (6) \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} (0) \cos nx dx \right]$$

This gives:

$$a_n = \frac{1}{\pi} \left[0 + \frac{6 \sin(nx)}{n} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 0 \right]$$

Recall,

$$\sin(-nx) = -\sin(nx)$$

$$a_n = \frac{12}{\pi n} \sin\left(n \frac{\pi}{2}\right)$$

If n is even, $a_n = 0$.

If n is odd,

For n = 1, 5, 9, 13,

$$a_n = \frac{12}{n\pi}$$

For n = 3, 7, 11, 15,

$$a_n = -\frac{12}{n\pi}$$

To find b_n :

$$b_n = \frac{1}{q} \int_{-q}^q f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

Since the function is even, $b_n = 0$.

By substituting the values into the Fourier series, we obtain:

$$f(x) = \frac{6}{2} + \frac{12}{\pi} \left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \frac{\cos 9x}{9} + \dots \right]$$

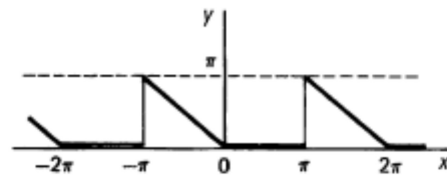
1.5 A FUNCTION THAT IS NEITHER EVEN OR ODD

A function could also not be even or odd. The Fourier series of the such function will have both the sine and cosine terms in its series.

QUESTION 3

Find the Fourier series of the function $f(x)$

$$f(x) = \begin{cases} 0 & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases}$$



Solution

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{q}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{q}\right)$$

$$a_0 = \frac{1}{q} \int_{-q}^q f(x) dx$$

$$a_n = \frac{1}{q} \int_{-q}^q f(x) \cos\left(\frac{n\pi x}{q}\right) dx$$

$$b_n = \frac{1}{q} \int_{-q}^q f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

Solution

$$q = \pi$$

$$a_0 = \frac{1}{q} \int_{-q}^q f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) dx = \frac{1}{\pi} \left[-\frac{x^2}{2} \right]_{-\pi}^0 = \frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{1}{q} \int_{-q}^q f(x) \cos\left(\frac{n\pi x}{q}\right) dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) \cos\left(\frac{n\pi x}{\pi}\right) dx \\ &= -\frac{1}{\pi} \left(\left[x \frac{\sin nx}{n} \right]_{-\pi}^0 - \frac{1}{n} \int_{-\pi}^0 \sin nx dx \right) = -\frac{1}{\pi} \left((0-0) - \frac{1}{n} \left[\frac{-\cos nx}{n} \right]_{-\pi}^0 \right) \\ &= -\frac{1}{\pi n^2} (1 - \cos n\pi) \end{aligned}$$

$$\cos n\pi = 1 \text{ for all even } n$$

$$\cos n\pi = -1 \text{ for all odd } n$$

Therefore,

$$a_n = -\frac{2}{\pi n^2} \text{ for odd } n \text{ and } a_n = 0 \text{ for even } n.$$

To calculate b_n

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) \sin nx dx \\ &= -\frac{1}{\pi} \int_{-\pi}^0 x \sin nx dx \\ &= -\frac{1}{\pi} \left\{ \left[x \left(\frac{-\cos nx}{n} \right) \right]_{-\pi}^0 + \frac{1}{n} \int_{-\pi}^0 \cos nx dx \right\} \\ &= -\frac{1}{\pi} \left\{ \frac{\pi \cos n\pi}{n} + \frac{1}{n} \left[\frac{\sin nx}{n} \right]_{-\pi}^0 \right\} = -\frac{\cos n\pi}{n} \end{aligned}$$

Therefore,

$$b_n = -\frac{1}{n} \text{ for all even } n$$

$$b_n = \frac{1}{n} \text{ for all odd } n$$

The Fourier series becomes:

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \dots \right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right)$$

1.6 SUMMARY ON EVEN, ODD AND NEITHER EVEN NOR ODD FUNCTIONS

- i) For even functions $f(x)$, the series will not contain any sine terms.
- ii) For odd functions $f(x)$, the series will not contain any cosine terms.
- iii) If the function is neither odd nor even, the series will, in general, contain a constant term, cosine terms and sine terms.

1.7 HARMONICS

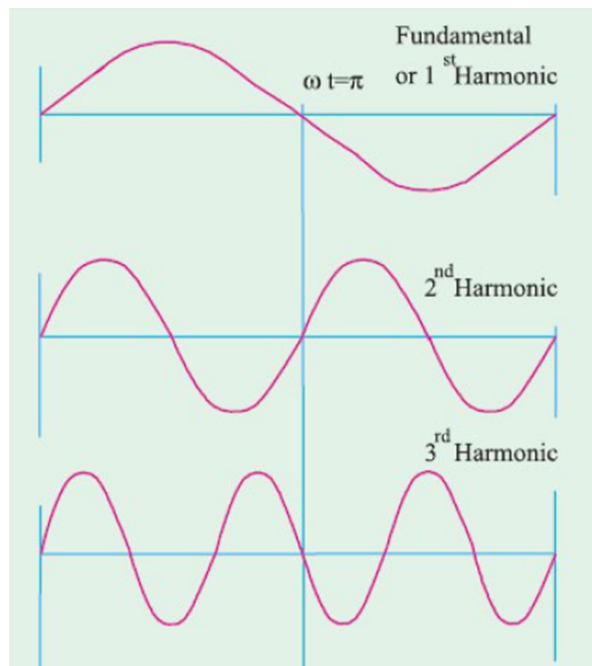
A function $f(x)$ is described as a series of a number of different sine components. The component with the largest period is the first harmonic or fundamental of $f(x)$.

$Y = A_1 \sin x$ is the first harmonic or fundamental

$Y = A_2 \sin 2x$ is the second harmonic

$Y = A_3 \sin 3x$ is the third harmonic

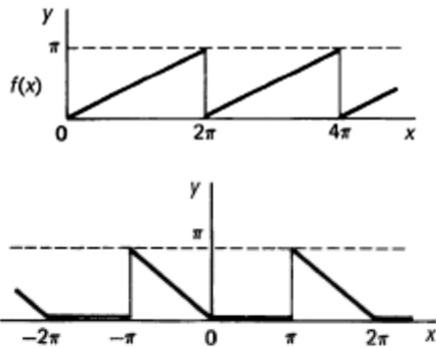
$Y = A_n \sin nx$ is the n th harmonic, with amplitude A_n and period $= 2\pi/n$



1.8 PERIODIC FUNCTIONS THAT ARE NON-SINUSODIAL

It should be noted that a function can be periodic without being obviously sinusoidal in appearance.

Examples of such functions are shown below

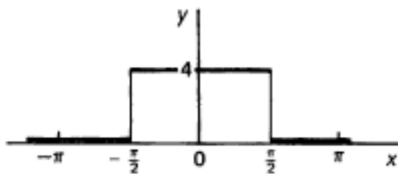


1.9 EFFECT OF HARMONICS

The complete representation of a function in Fourier series entails an infinite number of terms. The effects of including first few terms are shown for the function represented in Fourier series.

QUESTION 4

Obtain the Fourier series of the function $f(x)$ and show the effect of the inclusion of the first few terms on the function graphically.



$$f(x) = \begin{cases} 0 & -\pi < x < -\frac{\pi}{2} \\ 4 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

The period $T = 2\pi$

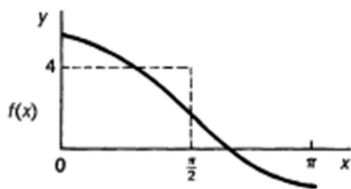
The Fourier series is:

$$f(x) = 2 + \frac{8}{\pi} \left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \dots \right]$$

Inclusion of terms and its effect graphically

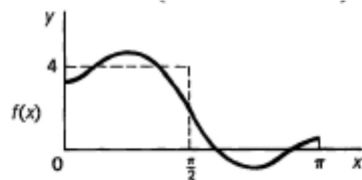
Stage 1

For $f(x) = 2 + \frac{8}{\pi} \cos x$



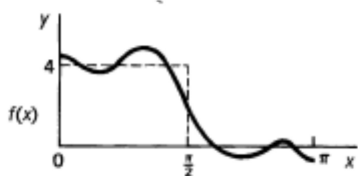
Stage 2

For $f(x) = 2 + \frac{8}{\pi} \left[\cos x - \frac{\cos 3x}{3} \right]$



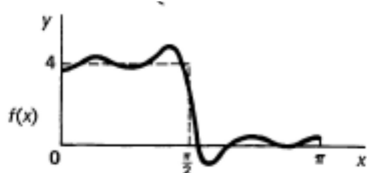
Stage 3

For $f(x) = 2 + \frac{8}{\pi} \left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} \right]$



Stage 4

For $f(x) = 2 + \frac{8}{\pi} \left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} \right]$

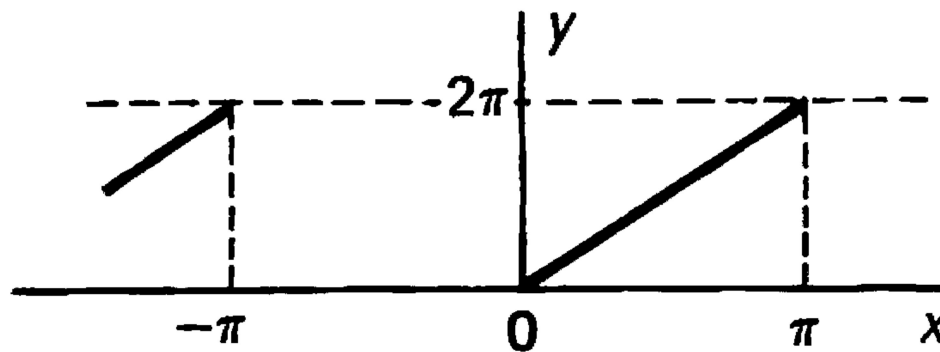


Note: It can be observed that as the number of included terms increases, the graph gradually takes the shape of the original square waveform.

1.10 HALF – RANGE SERIES

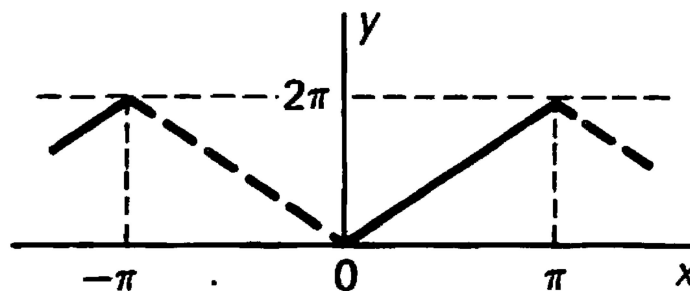
There are some situations where a function with period 2π is defined over a range of 0 to π as against $-\pi$ to π or 0 to 2π . If one is to proceed with obtaining the Fourier series, various approaches or assumptions will be made by various persons attempting to obtain a solution.

Let's take for example the function shown in the figure below. If we are told that between $x=0$ and $x = \pi$, $f(x) = 2x$ and noting that the period is 2π , there is no information on how the function behaves between $x = -\pi$ and $x = 0$.



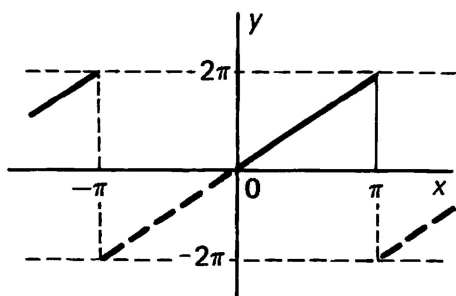
First approach:

If the waveform is constructed as shown in the graph below, the function will be an even function (the symmetry around the y-axis) and the Fourier series will only have cosine terms (including possibly a_0).



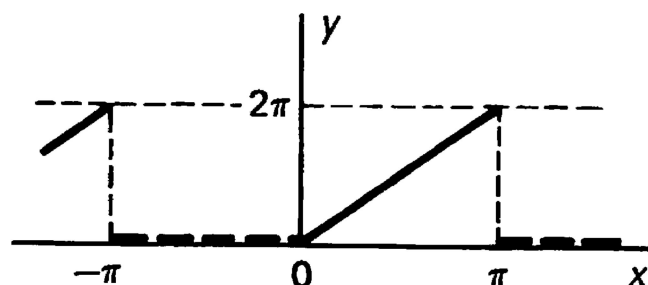
Second Approach

If the waveform is constructed as shown in the graph below, the function will be an odd function (the symmetry about the origin) and the Fourier series will only have sine terms.



Third Approach

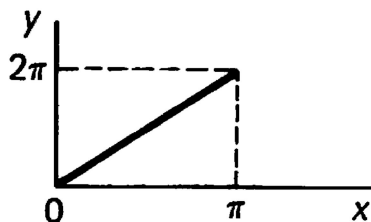
If something different from the earlier two are chosen between $x = -\pi$ and $x = 0$, then $f(x)$ will neither be even or odd and the Fourier series will contain a_0 , the cosine terms and the sine terms.



Note: The three approaches are all assumptions on how the function behaves between $x = -\pi$ and $x = 0$ but the Fourier series to be obtained will only apply to the defined region of the function $f(x)$ (i.e. between $x = 0$ and $x = \pi$). For this reason, series obtained are called HALF-RANGE SERIES.

QUESTION 4

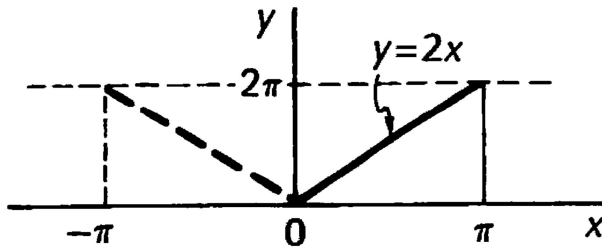
A function $f(x)$ is defined by $f(x) = 2x$ $0 < x < \pi$



$f(x+2\pi) = f(x)$. Obtain a half-range cosine series to represent the function.

SOLUTION

To obtain a half-range cosine series, an even function is required. The waveform will be symmetrical at the y-axis as shown in the figure below.



The Fourier coefficients are obtained. For half-range series

$$a_0 = \frac{1}{q} \int_{-q}^q f(x) dx = \frac{2}{q} \int_0^q f(x) dx = \frac{2}{\pi} \int_0^\pi 2x dx = \frac{2}{\pi} [x^2]_0^\pi = 2\pi$$

$$a_n = \frac{2}{q} \int_0^q f(x) \cos\left(\frac{n\pi x}{q}\right) dx = \frac{2}{\pi} \int_0^\pi 2x \cos nx dx = \frac{4}{\pi} \int_0^\pi x \cos nx dx$$

$$= \frac{4}{\pi} \left\{ \left[\frac{x \sin nx}{n} \right]_0^\pi - \frac{1}{n} \int_0^\pi \sin nx dx \right\}$$

$$= \frac{4}{\pi} \left\{ (0 - 0) - \frac{1}{n} \left[\frac{-\cos nx}{n} \right]_0^\pi \right\} = \frac{4}{\pi n^2} (\cos n\pi - 1)$$

$$\cos n\pi = 1 \quad \text{for even } n$$

$$\cos n\pi = -1 \quad \text{for odd } n$$

$$a_n = 0 \quad \text{for even } n$$

$$a_n = -\frac{8}{\pi n^2} \quad \text{for odd } n$$

To determine the value of b_n

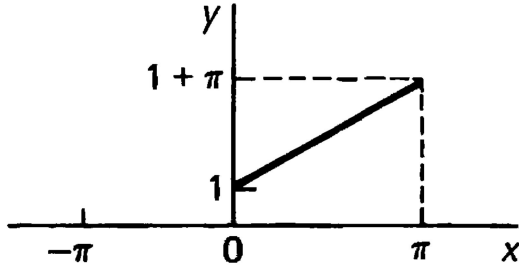
Since the function $f(x)$ is an even function, $b_n = 0$.

The Fourier series for the function becomes:

$$f(x) = \pi - \frac{8}{\pi} \left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \dots \right)$$

QUESTION 5

A function shown is shown graphically in the figure below. Obtain a half-range sine series to represent the function $f(x)$.



$$f(x) = 1 + x \quad 0 < x < \pi$$

$$f(x + 2\pi) = f(x)$$

SOLUTION

The waveform between $x = -\pi$ and $x = 0$ should be chosen in a way that the graph will be symmetric about the origin. This will make the function odd and the series will only contain sine terms.

Going by this, $a_0 = 0$ and $a_n = 0$, since the function is odd.

To calculate b_n :

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx = \frac{2}{\pi} \int_0^\pi (1+x) \sin nx dx = \frac{2}{\pi} \left\{ \left[(1+x) \frac{-\cos nx}{n} \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos nx dx \right\}$$

$$= \frac{2}{\pi} \left\{ -\frac{1+\pi}{n} \cos n\pi + \frac{1}{n} + \frac{1}{n} \left[\frac{\sin nx}{n} \right]_0^\pi \right\} = \frac{2}{\pi} \left\{ \frac{1}{n} - \frac{1+\pi}{n} \cos n\pi \right\}$$

$$b_n = \frac{2}{\pi n} [1 - (1+\pi) \cos n\pi]$$

$$\cos n\pi = 1 \quad \text{for even } n$$

$$\cos n\pi = -1 \quad \text{for odd } n$$

Therefore,

$$b_n = -\frac{2}{n} \quad \text{for all even } n$$

$$b_n = \frac{4+2\pi}{n\pi} \quad \text{for all odd } n$$

The Fourier series becomes

$$f(x) = \frac{4+2\pi}{\pi} \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right] - 2 \left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \dots \right]$$

